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On the solution of the Navier–Stokes equations for a spherically symmetric expanding flow

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It is shown that for low values of the ambient pressure the flow field for a steady spherically symmetric expansion can be divided into three parts termed the inviscid region, the intermediate layer and the shock layer. Analytic solutions are available in the first two regions and a complete integration of the equation is required in the third. Numerical solutions indicate that such a structure is achieved in the limit and the universality of the solutions in the individual regions is confirmed.

1. Introduction

The one-dimensional expansion of a gas in a vacuum or near vacuum is a well-known problem of gasdynamics familiar, in the textbook, as the flow in a Laval nozzle. The classical picture, when the viscosity is negligible, is of two possible inviscid flows, supersonic and subsonic, which may be characterized by their behaviour at infinity. The supersonic solution has finite velocity and zero temperature, density and pressure at infinity and the subsonic solution has zero velocity and finite temperature, density and pressure there. Any flow expanding from sonic conditions at some fixed radius must exhibit one of these variations at infinity provided that the neglected dissipative terms do not introduce a non-uniformity into the solution. The limiting solution of the Navier–Stokes equations for viscosity tending to zero introduces a further type of solution, however, which allows a discontinuous change through a normal shock wave from one of the above solutions to the other. The thermodynamics of the flow demands in an expanding flow that this jump shall be from the supersonic branch to the subsonic branch with an appropriate *increase* in entropy. These flows can thus describe flow into a complete vacuum using the supersonic branch in its entirety and flow into a region of finite pressure by either a continuous subsonic expansion or a supersonic expansion followed by a shock wave and subsonic expansion. The limiting structure of the flow has been studied extensively (Sakurai 1958) and a review of the limiting solutions has been given by the present author (Freeman 1970, subsequently referred to as I).

This description, relying as it does on the Navier–Stokes equations, requires that the gas be a continuous medium. The kinetic-theory description of the motion will not however be significantly different although the details of the shock structure and the transition from one solution to another will be changed. Recently, some progress has been made in the theory of expanding flows using the Boltzmann equation as the governing equation of the flow (see I) and it is therefore of some interest to attempt to extend the Navier–Stokes theory in the same way. To do this, the high Mach number limit of spherically symmetric expanding flow will be investigated. This limit is achieved physically when the ambient pressure is small but non-zero – a situation referred to as expansion into a near vacuum.

The thickness of the shock wave is governed by a reference Reynolds number based on the conditions at the sonic point or, more conveniently, the inverse of this quantity, which will be called α. Another non-dimensional ratio is the distance of the shock wave from the source, r_s, referred to the reference length r_*,† the distance of the sonic point from source. Classical theory assumes that r_s/r_* is of order one and α is small. It might be expected that non-uniformities in the theory might arise where r_*/r_s and α become comparable. This occurs when the shock wave approaches infinity, in regions where r, the distance from the source, is of order r_*/α. The condition that the shock wave is in this region may be derived from the ideal-gas relation

$$p = \rho RT,$$

where p is the pressure, ρ the density, T the temperature and R the gas constant, and the continuity relation

$$\rho u r^2 = \rho_* a_* r_*^2,$$

where u denotes the radial velocity and a the sound speed. We obtain

$$\frac{p}{p_*} = \frac{\rho}{\rho_*}\frac{T}{T_*} = \frac{a_*}{u}\frac{r_*^2}{r^2}\frac{T}{T_*}.$$

Thus, if the non-dimensional velocity u/a_* and temperature T/T_∞ remain of order unity, it is required that the non-dimensional pressure p/p_* be of order α^2 in this region. The limit is therefore achieved when the ambient pressure is inversely proportional to the Reynolds number squared. The region is then characterized by a flow in which the change in area, due to spherical symmetry, is of equal importance to the viscous dissipation. This might be termed a shock wave with area change or a *shock layer*. The full Navier–Stokes equations with spherical symmetry are required to describe the motion and, mathematically, a third-order ordinary differential equation or, more conveniently, a system of three first-order equations must be solved. It is the main purpose of this paper to describe the solution of these equations and the relationship of that solution to the whole flow field.

The manner in which this region is approached depends critically on the relationship assumed for the viscosity as a function of temperature. In this paper, it will be assumed that the viscosity is proportional to a power of the temperature and the exponent ω is less than one. In this case, it will be shown that prior to the region where $r_*/r = O(\alpha)$ described above a breakdown occurs in perturbations of the inviscid theory when $r_*/r = O(\alpha^\mu)$, where $\mu = [2\gamma - 1 - 2(\gamma-1)\,\omega]^{-1}$.‡ As γ, the ratio of specific heats, is greater than one, $\mu \leqslant 1$, equality being achieved when $\omega = 1$. For that particular case there is no distinction between the shock

† Suffix * refers to conditions at the sonic point.

‡ This exponent should not be confused with viscosity, which is usually denoted by μ.

layer and intermediate layer ($r_*/r = O(\alpha^\mu)$). For other values of ω, however, the shock layer is downstream of the intermediate layer.

Having stated the expected structure of the flow field, the mathematical limiting procedures used in the equations will now be developed.

2. The differential equations

The one-dimensional equations of motion of the flow of a perfect gas with spherical symmetry when non-dimensionalized with respect to conditions at the sonic point become

$$\alpha\left[\theta^\omega\left(\frac{d^2w}{dx^2}-\frac{2w}{x^2}\right)+\omega\theta^{\omega-1}\frac{d\theta}{dx}\left(\frac{dw}{dx}+\frac{w}{x}\right)\right]+\frac{dw}{dx}=\frac{1}{\gamma}\left\{\frac{d}{dx}\left(\frac{\theta}{w}\right)+\frac{2}{x}\frac{\theta}{w}\right\}, \tag{2.1}$$

$$\theta+\frac{\gamma-1}{2}\left(1+\frac{2\alpha\theta^\omega}{x}\right)w^2-\frac{\gamma+1}{2}=-\alpha\theta^\omega\left(\frac{3}{4\sigma}\frac{d\theta}{dx}+\frac{\gamma-1}{2}\frac{dw^2}{dx}\right), \tag{2.2}$$

where $x = r_*/r$, $\theta = T/T_*$ and $w = u/a_*$. Here r denotes radial distance, T temperature, u radial velocity, c sound speed and the suffix $*$ indicates values at the sonic point. The viscosity has been assumed proportional to T^ω and the Prandtl number σ is constant. The non-dimensional viscosity or inverse Reynolds number α is $\frac{4}{3}\nu_*/u_* r_*$, where ν denotes the kinematic viscosity. Equation (2.1) is the momentum equation and (2.2) is the energy equation with the density variation eliminated by using the continuity equation in the form

$$\rho u r^2 = \rho_* u_* r_*^2. \tag{2.3}$$

Our concern is with the limiting procedure $\alpha \to 0$. Applying this limit formally to (2.1) and (2.2) gives the inviscid-flow equations

$$w' = \frac{1}{\gamma}\left\{\left(\frac{\theta}{w}\right)'+\frac{2}{x}\frac{\theta}{w}\right\}, \tag{2.4}$$

$$\theta+\tfrac{1}{2}(\gamma-1)w^2 = \tfrac{1}{2}(\gamma+1), \tag{2.5}$$

where a prime denotes differentiation with respect to x.

These may be integrated to give

$$x^2/w\theta^{1/(\gamma-1)} = 1, \tag{2.6}$$

$$\theta+\tfrac{1}{2}(\gamma-1)w^2 = \tfrac{1}{2}(\gamma+1), \tag{2.7}$$

which may be recognized as entropy and energy equations. These equations give solutions corresponding to subsonic and supersonic flow which may be described by their behaviour as $x \to 0$ as

$$\theta \to 0, \quad w \to (\gamma+1)/(\gamma-1): \quad \text{supersonic}, \tag{2.8}$$

$$\theta \to \tfrac{1}{2}(\gamma+1), \quad w \to 0: \quad \text{subsonic}. \tag{2.9}$$

The nature of the transition from supersonic to subsonic flow through a shock wave including the limiting procedure has been described elsewhere (I) and will not be repeated here. Our main concern will be the problem associated with the double limit $x \to 0$ and $\alpha \to 0$.

The limiting behaviour of the solutions (2.6) and (2.7) on the supersonic branch may be obtained by a formal expansion procedure for small x as

$$\left.\begin{aligned} w &= w_0 + w_1 x^{2(\gamma-1)} + \dots, \\ \theta &= \theta_1 x^{2(\gamma-1)} + \dots, \end{aligned}\right\} \tag{2.10}$$

where

$$w_0^2 = \frac{\gamma+1}{\gamma-1}, \quad \theta_1 = \left(\frac{\gamma-1}{\gamma+2}\right)^{\frac{1}{2}(\gamma-1)}, \quad w_1 = -\frac{1}{(\gamma^2-1)^{\frac{1}{2}}}\left(\frac{\gamma-1}{\gamma+2}\right)^{\frac{1}{2}(\gamma-1)}.$$

Such a solution is not, however, a uniformly valid solution of (2.1) and (2.2). The relative magnitude of the next term in the expansion in α for x small is $\alpha x^{-\mu}$, where $\mu = [2\gamma - 1 - 2(\gamma-1)\omega]^{-1}$. This indicates that a new scaling must be sought in the region where $x = O(\alpha^{\mu})$.

Introducing new variables

$$W = \frac{w - w_0}{\alpha^{\lambda}}, \quad \Theta = \frac{\theta}{\alpha^{\lambda}}, \quad X = \frac{x}{\alpha^{\mu}},$$

where $\lambda = 2(\gamma-1)\mu$, the equations become in the limit $\alpha \to 0$

$$\omega w_0 \frac{\Theta' \Theta^{\omega-1}}{X} + \frac{dW}{dX} - \frac{2w_0 \Theta^{\omega}}{X^2} = \frac{1}{\gamma W_0}\left[\frac{d\Theta}{dX} + \frac{2}{X}\Theta\right], \tag{2.11}$$

$$\Theta + w_0(\gamma-1)\left[W + \frac{w_0\Theta^{\omega}}{X}\right] = 0. \tag{2.12}$$

These equations can be readily solved to give

$$\Theta = \left\{\frac{w_0^2\gamma(\gamma-1)(1-\omega)}{[2\gamma-1-2\omega(\gamma-1)]X} + \theta_1^{1-\omega} X^{2(\gamma-1)(1-\omega)}\right\}^{1/(1-\omega)}, \tag{2.13}$$

$$W = -\frac{\Theta}{w_0(\gamma-1)} - \frac{w_0\Theta^{\omega}}{X}, \tag{2.14}$$

where the matching condition from (2.10) has been used to evaluate the arbitrary constant.

It will be assumed that $\omega \leqslant 1$ and hence $\mu \leqslant 1$. The behaviour for $\omega > 1$ gives a singularity in the region of validity of (2.13) and the whole structure becomes different. The particular case $\omega = 1$ gives

$$\Theta = \theta_1 X^{2(\gamma-1)} \exp[\gamma(\gamma+1)/X], \tag{2.15}$$

but as has already been noted, this result is only of academic interest since then the region of validity of (2.15) is swallowed up by the shock-layer behaviour.

The matching of (2.13) and (2.14) with the inviscid solution upstream has already been achieved. Downstream the behaviour of (2.13) corresponds to an algebraic increase $\Theta \sim X^{-1/(1-\omega)}$ as $X \to 0$. Rewritten in terms of the original variables this gives $\theta \to (x/\alpha)^{-1/(1-\omega)}$ and thus θ remains of order one in a region of thickness α. Similarly, an examination of (2.14) shows that W remains of order one in this region. The shock-layer behaviour is thus described in terms of a boundary-layer region of thickness α which is 'inside' the intermediate layer of thickness α^{μ}, where $\mu < 1$. Within this layer, θ and w remain of order one.

3. The shock-layer solution

To rescale the variables to conform to the suggested magnitudes in the far field, we put $Y = x/\alpha$ and retain θ and w as order one variables. Substitution in (2.1) and (2.2) then gives

$$\theta^{\omega}\left(\frac{d^2w}{dY^2}-\frac{2w}{Y^2}\right)+\omega\theta^{\omega-1}\frac{d\theta}{dY}\left(\frac{dw}{dY}+\frac{w}{Y}\right)+\frac{dw}{dY}=-\frac{1}{\gamma}\left\{\frac{d}{dY}\left(\frac{\theta}{w}\right)+\frac{2}{Y}\frac{\theta}{w}\right\} \tag{3.1}$$

and

$$\theta+\frac{\gamma-1}{2}\left(1+\frac{2\theta^{\omega}}{Y}\right)w^2-\frac{\gamma+1}{2}=\theta^{\omega}\left(\frac{3}{4\sigma}\frac{d\theta}{dY}+\frac{\gamma-1}{2}\frac{dw^2}{dY}\right). \tag{3.2}$$

These are the full Navier–Stokes equations in a scaled form, indicating that no significant simplification can be achieved in this region. The equations describe a shock wave structure including an area change. Asymptotically the solutions of (3.1) and (3.2) must match for $Y \to \infty$ with the asymptotic limit of (2.13) and (2.14) as $X \to 0$. Downstream we expect that the characteristic behaviour associated with the subsonic branch of the inviscid solution will be obtained asymptotically as $Y \to 0$. This implies that $\theta \to \frac{1}{2}(\gamma+1)$ and $w \sim W_0 Y^2$. Introducing a non-dimensional pressure P, then

$$P = x^2\theta/w = \alpha^2 Y^2\theta/w \tag{3.3}$$

and as $Y \to 0$

$$P \to P_0 = \frac{\alpha^2}{W_0}\left(\frac{\gamma+1}{2}\right). \tag{3.4}$$

Thus W_0 represents the inverse pressure variation at infinity and the pressure itself as already observed is proportional to an inverse Reynolds number squared in this region.

It is obviously not possible to obtain analytical solutions of (3.1) and (3.2) and any further progress must be made numerically. It has already been shown in I that the scaling described above enables numerical results obtained by direct integration of (3.1) and (3.2) (Gusev & Zhubakova 1969; Rebrov & Chekmaryov 1970) to be correlated. Such results were not, however, derived with the intention of checking the above theory and thus have serious shortcomings if used for this purpose. The main difficulties are that Gusev & Zhubakova (1969) are only concerned with $\omega = 1$ and Rebrov & Chekmaryov (1970) choose only to use the Sutherland viscosity law. It is clearly necessary to derive numerical results directly applicable to the equations as described above if any significant comparisons are to be made between the numerical and analytic structures of the solutions. A numerical procedure similar to that described by Gusev & Zhubakova (1969) was thus developed.

4. Numerical method and results

One of the main difficulties in obtaining numerical solutions to asymptotic theories similar to that described above is that the boundary conditions given for the equations are usually of an asymptotic nature. These conditions represent singular solutions of the differential equation and consequently are very special in their behaviour. It is important, however, only to consider solutions close to these solutions since spurious integral curves abound in problems of this kind. Since the system is of third order it is not possible to do any general analysis of the nature of the singular points of the differential equation and it becomes necessary to use elaborate numerical techniques to extract the solutions of physical interest. To proceed with a conventional integration scheme requires three initial conditions, on say θ, w and dw/dx, and it is the choice of these three parameters which are of major importance. Preliminary calculations indicated that if these were not chosen correctly integral curves soon reached regions of the function space where the solutions were unrealistic – with temperature becoming negative, for example.

To overcome these difficulties it was necessary to isolate the region of the space of θ, w and dw/dx where realistic solutions could be found by accurately computing the asymptotic behaviour of the solutions required and using only values in the neighbourhood of these values to start the integration procedure. Early attempts to evaluate the terms of the asymptotic expansion proved inadequate for this task and it was necessary to develop a technique whereby the evaluation of the terms of the asymptotic series could be done on the computer. In this way twenty to thirty terms of the expansion could be computed without difficulty and only then was it possible to develop a satisfactory integration scheme. The method employed was to start the integration at different points in the range using initial values determined from the asymptotic expansion at $Y = 0$.

The choice of starting-point for the integration from the asymptotic values gives a variety of integral curves but too low an initial value gives curves which have negative values of temperature and are thus physically unrealistic. Such curves were in fact obtained by Gusev & Zhubakova (1969). For larger values of Y than these, the curves can be continued back to sonic conditions. This enables a value of α to be assigned to them by noting the value of Y at which the sonic value of temperature and density is reached. An added check on the correctness of this procedure is the fact that these values are reached simultaneously to a high degree of accuracy. Each integration is done for a particular value of the downstream pressure by specifying W_0. The boundary between the physically unrealistic curves described above and the realistic curves which can be continued to the sonic point is thus seen to be the limiting curve for $\alpha = 0$ required in the shock layer, since, as the unrealistic curves are approached, the value of α reduces. A further check on the correctness of the integral curves may be made by comparing the value of α required to give the correct scaling in the intermediate region with that computed from the sonic point. In practice, this value may be most readily computed by a scaling of the minimum temperature reached. This value is directly proportional to the temperature. The temperature variation computed by integrations of this kind is extremely close to that predicted by the preceding analysis. Numerically, the changes in the value of the non-dimensional temperature are very dramatic, remaining of order one in the inviscid region and shock layer and becoming exceedingly small over a significant range in the intermediate layer as expected because of the smallness of α. The

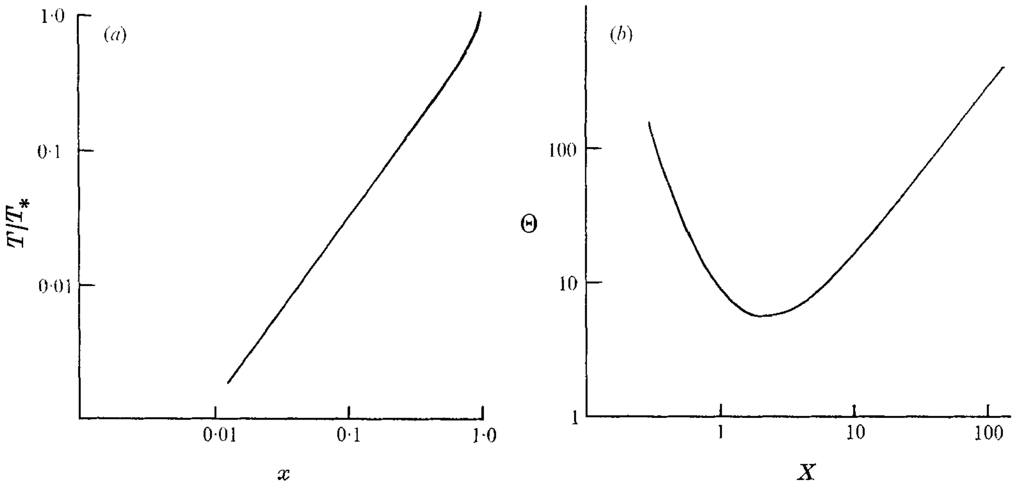

FIGURE 1. Temperature variation in (a) inviscid region and (b) intermediate layer.

rapidity with which these changes in temperature occur as the various regions are approached causes considerable numerical problems, however. No attempt was made to use sophisticated integration schemes – a fourth-order Runge–Kutta scheme was used throughout – but exceedingly small integration steps were required if the integration scheme was to proceed satisfactorily. A consequence of this was the extremely long runs which were required on the Newcastle University IBM 360/67 computer. Even after satisfactory starting values had been established runs of 30 min were necessary. It is possible, of course, that a program could be developed to improve upon this. Use of the on-line facility of the computer allowed successive integral curves to be computed for smaller and smaller values of α. The smallest value of α achieved was below 5×10^{-6} and this required a knowledge of the starting value of Y correct to 12 figures. To complete such runs would, however, have been prohibitive in terms of computer time. Such low values of α are not required to plot the solutions within the accuracy of the figures given below. The main results were obtained for $\gamma = \frac{5}{3}$, $\sigma = \frac{3}{4}$ and $\omega = \frac{3}{4}$. No major differences were observed for computations at other values of the parameters and only these values are used in the results discussed below.

The results are given in the form of a series of curves which are plotted with similar logarithmic scales both horizontally and vertically. These scales extend over many cycles so the physical magnitude of the variables is to some extent obscured and the large variation in numerical values, which would be immediately evident on a linear plot, obliterated. The advantage of using such a plot is, however, that any algebraic scaling of the variables simply shifts the curves by a constant amount vertically or horizontally.

With this in mind, figures 1 (a) and (b) show the variation of temperature as given by equations (2.6) and (2.7) in the inviscid region and equation (2.13) in the intermediate layer, respectively.

A typical integration in the far field is shown in figure 2; extending the integration back to the sonic values gives $\alpha = 0{\cdot}00042$ with $W_0 = 0{\cdot}75$. The scales of

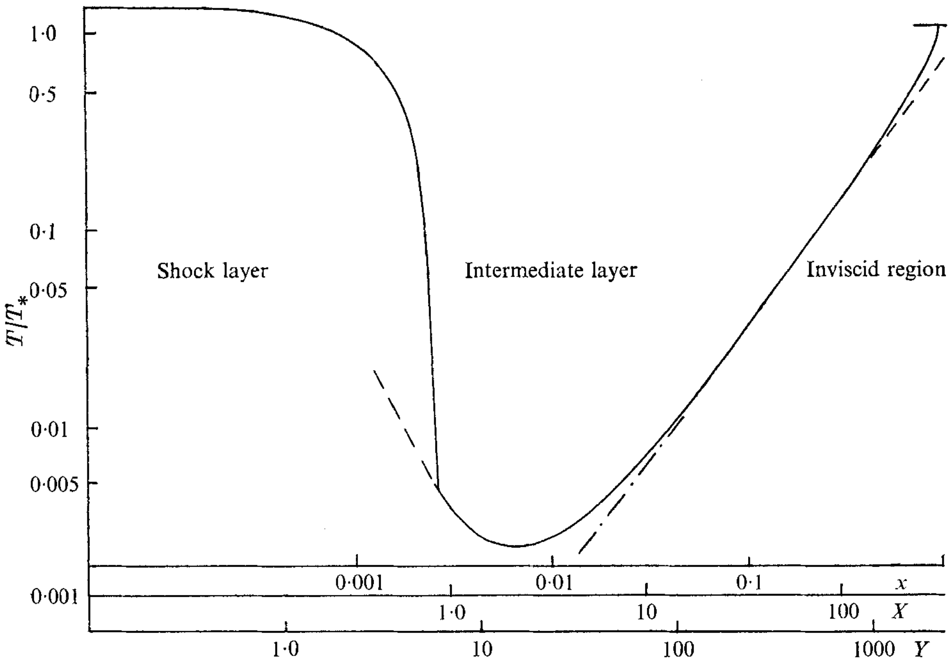

FIGURE 2. Temperature variation for $W_0 = 0{\cdot}75$ and $\alpha = 0{\cdot}00042$.
— —, intermediate-layer solution; —·—·, inviscid solution.

the independent variables in the three regions are plotted along the abscissae. It is comforting to observe that they remain of order one in their appropriate regions of usefulness. Within the accuracy of the display the analytic behaviour of temperature in the inviscid and intermediate layers is identical with the computed values, only deviating outside the region of validity as indicated by the dotted curves. As noted above, the use of the logarithmic scales tends to obscure the extremely wide variation of the magnitude of the temperature ratio. It may be observed in figure 2, for example, that the temperature remains at less than half a per cent of its sonic value for a range of Y from 6 to 70.

In figure 3, a comparison of two curves for fixed downstream pressure (i.e. W_0) and varying α are shown. On the scales used it is clear that the variations in the intermediate layer and inviscid region are identical, only changing by the constant relative shift due to the differing values of α.

The effect of variation of downstream pressure (W_0^{-1}) is, as expected, to change the value of the temperature in the shock layer and, for fixed α, to leave unaffected the distribution in the intermediate and inviscid regions.

Using the values of the temperature obtained from these integrations it is possible to derive the asymptotic curves required in the shock-layer region by extrapolation. These curves are indicated in figure 4 for different values of W_0. A curious feature of these curves is their insensitivity to increasing W_0 over most of the range. It is also difficult to show the matching to the upstream layer since this occurs at such low values of θ.

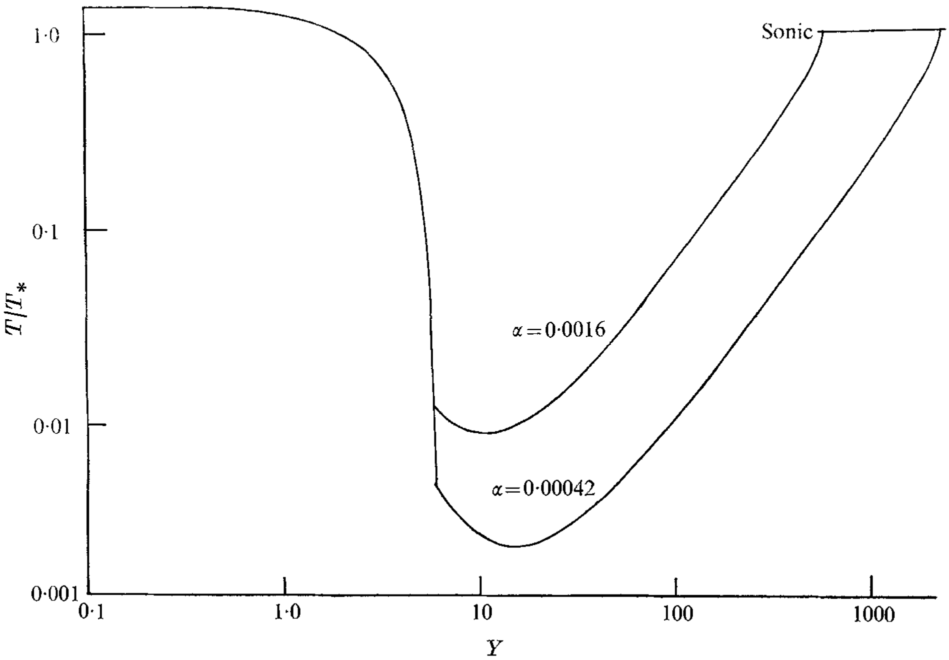

FIGURE 3. Temperature variation for $W_0 = 0{\cdot}75$ and two values of α.

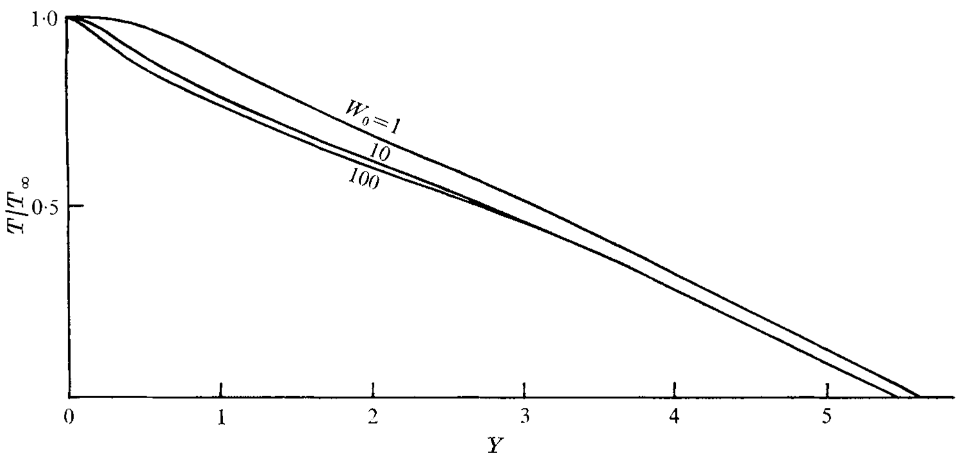

FIGURE 4. Variation of temperature in shock layer for varying ambient pressure (linear scales).

5. Conclusion

The structure of the flow field for expanding spherically symmetric flow of a perfect gas into a near vacuum has been investigated both analytically and numerically. Since the shock-layer solution required integration of the complete equations of motion, it has been possible to verify numerically that the scaling suggested by the asymptotic limiting process $\alpha \to 0$ when the ambient pressure is of order α^2 is in fact achieved. The universality of the limiting solutions in their three respective regions of validity has thus been demonstrated.

As was described in I, the corresponding results for the Boltzmann equation are already available for the inviscid region and the intermediate layer, although the matching process between these regions and the far field is more complex. Some effort has recently been made to compute the flow in this region, however, (Thomas 1972) and more detailed computations for the particular model gas used in this paper (i.e. using the appropriate viscosity variation with temperature or, in kinetic-theory terms, the appropriate variation of collision cross-section with temperature) might enable a similar structure to that found here to be verified by numerical computation.

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